Quasi-Exact Solvability and Deformations of $Sl(2)$ Algebra

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Abstract. Algebraic structure of a class of differential equations including Heun is shown to be related with the deformations of $sl(2)$ algebra. These include both quadratic and cubic ones. The finite dimensional representation of cubic algebra is explicitly shown to describe a quasi-exactly solvable system, not connected with $sl(2)$ symmetry. Known finite dimensional representations of $sl(2)$ emerge under special conditions. We answer affirmatively the question raised by Turbiner: "Are there quasi-exactly solvable problems which can not be represented in terms of $sl(2)$ generators?" and give the explicit deformed symmetry underlying this system.

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1. Introduction

It is well known that, only a few one dimensional Schrödinger eigenvalue problems exhibit exact solvability [1]. Recently, the class of spectral problems showing quasi-exact solvability has attracted considerable attention [2]. Quasi-exactly solvable (QES) systems have partially tractable energy spectrum. Only a few eigenvalues and their eigenfunctions are analytically approachable [3, 4, 5]. They have a deep connection with finite dimensional representations of $sl(2)$ group and are also connected with equilibrium electrostatic configurations [3]. Interestingly, a quasi-exactly solvable system has been identified, which has connection with Heun differential equation [6] and is not amenable to the $sl(2)$ based classification [7].

In this paper, we analyze the algebraic structure of a wide class of differential equations, including Heun as a subsystem. It is found that the underlying symmetries of this class of equations are deformed $sl(2)$. These deformations are of cubic and quadratic type for Heun and confluent or bi-confluent Heun differential equations, respectively. It is also found that, the finite dimensional representation of cubic algebra, describes a quasi-exactly solvable system, not connected with $sl(2)$ symmetry.

The paper is organized as follows. In the following section, we will consider a class of differential equations with regular singularities ranging from 0 to 3, of which Heun is

a subclass. We identify the operators in the respective differential equations, from which the deformation of the algebra emerges. We also compute the Casimir characterizing the representations. Subsequently, we consider the exact and quasi-exact solvability of the differential equations, when related to appropriate eigenvalue equations. The respective conditions of exact and quasi-exact solvability produces Hypergeometric and a type of Heun differential equation. Finite dimensional representation of $sl(2)$ algebra is shown to emerge under certain conditions, which characterize a number of quasi-exactly solvable systems. Our analysis, not only answers positively the question raised by Turbiner [3] : "Are there quasi-exactly solvable problems which can not be represented in terms of $sl(2)$ generators?", but also gives the explicit deformed symmetry underlying this system.

Table 1. The parameters for the Heun class and the Jacobi differential equations and their related symmetries.

Differential equations	Values of parameters $a_i (i = 0, \ldots, 8)$								Symmetry algebra
	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	
Heun Equation	1	$-(c+1)$	c	0	$\gamma+\delta+\varepsilon$	$-[\gamma(c+1)+\delta c+\varepsilon)]$	γc	$\alpha\beta$	$-q$	*Cubic*
Confluent Heun	0	1	-1	0	ν	$\gamma+\delta-\nu$	$-\gamma$	$\alpha\nu$	$-\sigma$	*Quadratic*
Bi-Confluent Heun	0	0	1	0	-2	$-\beta$	$\alpha+1$	$\gamma-\alpha-2$	$-\frac{1}{2}[\delta+(\alpha+1)\beta]$	*Quadratic*
Doubly Confluent	0	1	0	0	-1	τ	ν	$-\alpha$	q	*Linear*
Jacobi	0	-1	0	1	0	$-(\alpha+\beta+2)$	$\beta-\alpha$	0	$n(n+\alpha+\beta+1)$	*Cubic*

2. Heun Class of Differential Equations and Deformed $Sl(2)$ Algebra

As mentioned before, the spectral Schrödinger equation,

$$\hat{H}\psi = E\psi, \hat{H} = -\frac{d^2}{dx^2} + V(x), \qquad x \in (-\infty, +\infty) \text{ or } x \in [0, \infty) \tag{1}$$

can be connected with differential equations, having different singularity structure by suitable change of variables and appropriate point canonical transformations [8]. These differential equations are generically of the form,

$$\left[f_1(x)\frac{d^2}{dx^2} + f_2(x)\frac{d}{dx} + f_3(x)\right]\psi(x) = 0, \tag{2}$$

where

$$f_1(x) = a_0x^3 + a_1x^2 + a_2x + a_3, \tag{3a}$$
$$f_2(x) = a_4x^2 + a_5x + a_6, \tag{3b}$$
$$f_3(x) = a_7x + a_8; \tag{3c}$$

with $a_i \in \mathbf{R}$ for $i = 0, \ldots, 8$. These equations have regular singularities, varying between 0 to 3, depending on the values of a_i's.

Keeping in mind quasi-exactly solvable problems and their connection with $sl(2)$ algebra, we now start with a finite dimensional representation of $sl(2)$ algebra with spin j, in a space of monomials x^{j+m} $(m \leq |j|)$. The following generators

$$J^+ = x^2\frac{d}{dx} - 2jx, \qquad J^0 = x\frac{d}{dx} - j, \qquad J^- = \frac{d}{dx}, \tag{4}$$

satisfy the closed algebra

$$[J^+, J^-] = -2J^0, \qquad [J^0, J^\pm] = \pm J^\pm. \tag{5}$$

The fact that the representation space is finite and algebraically interrelated explains the quasi-exact solvability of the corresponding Schrödinger equation [9].

To analyze the symmetry of the QES problems for the aforementioned class of differential equations (2), we introduce a set of operators $\{P_+, P_0, P_-\}$ in the space of monomials such that :

$$P_+x^n = c_+x^{n+1}, \qquad P_0x^n = c_0x^n, \qquad P_-x^n = c_-x^{n-1}.$$

The general differential equation (2) can be cast in terms of $\{P_+, P_0, P_-\}$ operators if $a_3 = 0$:

$$[P_+ + F(P_0) + P_-]\,\psi(x) = 0. \tag{6}$$

Clearly

$$P_+ = a_0x^3\frac{d^2}{dx^2} + a_4x^2\frac{d}{dx} + a_7x, \tag{7a}$$

$$F(x\frac{d}{dx}) = a_1x^2\frac{d^2}{dx^2} + a_5x\frac{d}{dx} + a_8, \qquad P_0 = x\frac{d}{dx} - j, \tag{7b}$$

$$\text{and} \qquad P_- = a_2x\frac{d^2}{dx^2} + a_6\frac{d}{dx}. \tag{7c}$$

$F(x\frac{d}{dx})$ can be simplified in terms of P_0:

$$F(P_0) = a_1 P_0^2 + ((2j-1)a_1 + a_5)P_0 + (a_1 j^2 - (a_1 - a_5)j + a_8).$$

The algebraic structure can be inferred from the following closed form,

$$[P_+, P_-] = \alpha_1 P_0^3 + \beta_1 P_0^2 + \gamma_1 P_0 + \delta_1 = f(P_0) \qquad \text{and} \qquad [P_0, P_\pm] = \pm P_\pm \quad (8)$$

where,

$$\alpha_1 = -4a_0a_2, \qquad \beta_1 = 6(1-2j)a_0a_2 - 3a_2a_4 - 3a_0a_6$$
$$\gamma_1 = (-2a_0a_2 + 3a_0a_6 - 2a_4a_6 - 2a_2a_7 + a_2a_4)$$
$$+2(6a_0a_2 - 3a_2a_4 - 3a_0a_6)j + (-12a_0a_2)j^2,$$
$$\delta_1 = -a_6a_7 + (-2a_0a_2 + 3a_0a_6 - 2a_4a_6 - 2a_2a_7 + a_2a_4)j$$
$$+(6a_0a_2 - 3a_2a_4 - 3a_0a_6)j^2 + (-4a_0a_2)j^3.$$

The eigenvalue of the Casimir operator, $C = J^-J^+ + g(J^0)$ is (a_6a_7), with $g(J^0) - g(J^0 - 1) = f(J^0)$ [10] for the present case. Hence, the constituent operators of the differential equation (2) satisfy the *cubic deformation* of the $sl(2)$ structure [11, 12, 13, 14]. Some well known differential equations [6, 15, 16, 17, 18, 19] with the above structure, are listed in Table-1. The solutions to the above equations can be obtained employing a recent approach, which connects the space of monomials to the solution space [20]. For this purpose, the differential equation should be cast in the form (6) provided the condition $F(x\frac{d}{dx})x^\lambda = 0$. This leads to,

$$\lambda_\pm = \frac{1}{2a_1}\left[-(a_5 - a_1) \pm \sqrt{(a_5 - a_1)^2 - 4a_1a_8}\right]. \qquad (9)$$

The solution to the differential equation (2) can be given by,

$$\psi(x) = C_{\lambda_\pm} \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{(D + \lambda_+)(D + \lambda_-)}\{P_+ + P_-\}\right]^m x^{-\lambda_\pm}. \qquad (10)$$

The above is exactly solvable only when $P_- = 0$ and is quasi-exactly solvable under certain conditions, when both P_+ and P_- are present [5].

3. Exact and Quasi-Exact Solvability

It has been shown that [3], quasi-exactly-solvable Schrödinger equation that are connected to bilinear representation of $sl(2)$, can be cast in the differential form:

$$-P_4(x)\frac{d^2\varphi}{dx^2} + P_3(x)\frac{d\varphi}{dx} + (P_2(x) - \varepsilon)\varphi = 0, \qquad (11)$$

where P_i's are the polynomials of the i-th power $(i = 2, 3, 4)$:

$$P_4 = a_{++}x^4 + a_{+0}x^3 + (a_{+-} + a_{00})x^2 + a_{0-}x + a_{--},$$
$$P_3 = 2(2j-1)a_{++}x^3 + [(3j-1)a_{+0} + b_+]x^2$$
$$+[2j(a_{+-} + a_{00}) + a_{00} + b_0]x + ja_0 + b_-,$$
$$-P_2 = 2j(2j-1)a_{++}x^2 + 2j(ja_{+0} + b_+)x + a_{00}j^2 + b_0j$$

We now investigate the conditions for exact and quasi-exact solvability of the aforementioned class of spectral problems. Comparing this equation with (2) yields $a_{++} = a_{--} = 0$ and leads to,

$$-a_{+0} = a_0,\ -(a_{+-} + a_{00}) = a_1,\ -a_{0-} = a_2,$$
$$2j(a_{+-} + a_{00}) + a_{00} + b_0 = a_5,\ ja_{0-} + b_- = a_6,$$
$$-2j(ja_{+0} + b_+) = a_7,\ -(a_{00}j^2 + b_0 j) = a_8.$$

The conditions that are to be imposed upon the differential equation (2), for exact-solvability and quasi-exact solvability are :

$$a_{++} = a_{+0} = b_+ = 0 \tag{12}$$

and

$$a_{0-} = a_{--} = b_- = 0, \tag{13}$$

respectively.

The exact solvability conditions (12), applied to (2), yield

$$x\left(x + \frac{a_2}{a_1}\right)\frac{d^2\psi(x)}{dx^2} + \frac{a_5}{a_1}\left(x + \frac{a_6}{a_5}\right)\frac{d\psi(x)}{dx} + \frac{a_8}{a_1}\psi(x) = 0. \tag{14}$$

which is the *Hypergeometric* differential equation. The two solutions can be cast as a mapping connecting the monomial space to the space of polynomials [21] :

$$\psi_{\lambda_\pm}(x) = C_{\lambda_\pm} exp\left[\frac{1}{(D + \lambda_\mp)}\left(a_2 x\frac{d^2}{dx^2} + a_6\frac{d}{dx}\right)\right]x^{-\lambda_\pm}. \tag{15}$$

The quasi-exact solvability conditions given in Table 2, produces a differential equation of the *Heun* type

$$x^2\left(x + \frac{a_1}{a_0}\right)\frac{d^2\psi(x)}{dx^2} + \frac{a_4}{a_0}x\left(x + \frac{a_5}{a_4}\right)\frac{d\psi(x)}{dx} + \frac{a_7}{a_0}\left(x + \frac{a_8}{a_7}\right)\psi(x) = 0. \tag{16}$$

The two solutions to the above differential equation are [22]

$$\psi_{\lambda\pm}(x) = C_{\lambda_\pm} exp\left[\frac{(-1)}{(D + \lambda_\mp)}\left(a_0 x^3\frac{d^2}{dx^2} + a_4 x^2\frac{d}{dx} + a_7 x\right)\right]x^{-\lambda_\pm}. \tag{17}$$

Under certain conditions, these series can terminate yielding polynomial solutions, that leads to known QES systems [5].

Table 2. Quasi-exact solvability conditions

Condition	Result	Implication
$a_{0-} = 0$	$a_2 = 0$	Linear term in $f_1(x)$ is zero
$b_- = 0$	$a_6 = 0$	Constant term in $f_2(x)$ is zero

We now deal with the Heun equation, not amenable to the $sl(2)$ structure. We answer Turbiner's question affirmatively, by exploiting the algebraic symmetry, through a physical example.

Let us consider the equation

$$\left(a\frac{d^2}{d\sigma^2}+b\frac{d}{d\sigma}+c+{\nu_n}^2\right)\psi_n=0 \tag{18}$$

where

$$a=(1-\sigma^2)^2(\sigma^2+\epsilon^2), \qquad b=\sigma(1-\sigma^2)(1-2\epsilon^2-3\sigma^2),$$
$$c=-1+2\epsilon^2+6\sigma^2(2-\epsilon^2)-15\sigma^4 \qquad \text{and} \qquad \nu^2=4(1+\epsilon^2)\ {\omega_n}^2/\mu^2.$$

This equation arises from a Schrödinger like equation, when one performs stability analysis around a kink solution of the ϕ^6 theory in one dimension [23]. By defining $\zeta=\sigma^2$, $\psi_n=(1-\zeta)^s f$ where $s=\left[1-\left(\frac{\omega_n}{\mu}\right)^2\right]^{\frac{1}{2}}$, (18) can be cast into the canonical form of the Heun equation:

$$\frac{d^2f}{d\zeta^2}+\left[\frac{\gamma}{\zeta}+\frac{\delta}{(\zeta-1)}+\frac{\varepsilon}{(\zeta-a)}\right]\frac{df}{d\zeta}+\frac{\alpha\beta\zeta-q}{\zeta(\zeta-1)(\zeta-a)}f=0, \tag{19}$$

where

$$\gamma=\varepsilon=\frac{1}{2},\delta=1+2s,a=-\epsilon^2,$$
$$\alpha=\left(-\frac{5}{2}-s\right),\beta=\left(\frac{3}{2}-s\right),$$
$$\text{and} \qquad q=-\frac{1}{4}\left[1-2\epsilon^2-4(1-s^2)(1+\epsilon^2)+2s\epsilon^2\right].$$

The Heun equation is known to arise in several other contexts of physical interest [19, 24, 25, 26, 27]. Equation (19) can be written in the form,

$$(\hat{H}-q)f=\{P_+-aP_-+F(P_0)\}f=0, \tag{20}$$

where

$$P_+=\zeta^3\frac{d^2}{d\zeta^2}+(\gamma+\delta+\varepsilon)\zeta^2\frac{d}{d\zeta}+\alpha\beta\zeta, \tag{21a}$$

$$P_-=\zeta\frac{d^2}{d\zeta^2}+\gamma\frac{d}{d\zeta}, \tag{21b}$$

$$P_0=\zeta\frac{d}{d\zeta}-j, \tag{21c}$$

$$\text{and} \qquad F(P_0)=n_2P_0^2+n_1P_0+n_0. \tag{21d}$$

Here,

$$n_2=-(a+1), \qquad n_1=\{a-(\gamma(a+1)+a\delta+\varepsilon)+1\}-2j(a+1),$$
$$n_0=-(a+1)j^2+\{a-(\gamma(a+1)+a\delta+\varepsilon)+1\}j-q.$$

Here P_+ and P_- are raising and lowering operators in the space of monomials $\{1, \zeta, \zeta^2, \ldots, \zeta^{2j}\}$, where $N = (2j+1)$ is the dimension of the space. The closed algebra is a *cubic* deformation of $sl(2)$,

$$[P_+, P_-] = \alpha_1 P_0^3 + \beta_1 P_0^2 + \gamma_1 P_0 + \delta_1 \ , \qquad [P_0, P_\pm] = \pm P_\pm \tag{22}$$

with

$$\begin{aligned}
&\alpha_1 = 4, \qquad \beta_1 = 3(2\gamma + \delta + \epsilon - 2) + 12j, \\
&\gamma_1 = \{2\alpha\beta - 3\gamma + 2 + (2\gamma - 1)(\gamma + \delta + \epsilon) \\
&+6(2\gamma + \delta + \epsilon - 2)j + 9j^2\}, \\
&\delta_1 = \alpha\beta\gamma + \{2\alpha\beta - 3\gamma + 2 + (2\gamma - 1)(\gamma + \delta + \epsilon)\}j \\
&+3(2\gamma + \delta + \epsilon - 2)j^2 + 4j^3.
\end{aligned}$$

We now assume a polynomial solution of (19) of the form,

$$f(\zeta^l) = g_1 \zeta^l + g_2 \zeta^{2l} + g_3 \zeta^{3l} + \ldots$$

where $l = 1$ or $l = \frac{1}{2}$, correspond to the independent variable σ^2 and σ respectively in (18). Thus, to obtain a polynomial solution of definite degree, we terminate the polynomial at ζ^{n-1} and impose $P_+ \zeta^{n-1} = 0$. This leads to

$$s^2 + (2n-1)s + n^2 - n - \frac{15}{4} = 0. \tag{23}$$

The above restriction allows, $n = 2, s = 1/2$ and $n = 3/2, s = 1$, yielding,

$$\psi(x)_{n=2} = (1-\zeta)^{1/2} f(\zeta) = \left(\frac{\frac{\epsilon^2+1}{\epsilon^2}}{\frac{\epsilon^2+1}{\epsilon^2} + \sinh^2(\frac{\mu x}{2})}\right)^{1/2} \frac{\sinh^2(\frac{\mu x}{2})}{\left(\frac{\epsilon^2+1}{\epsilon^2} + \sinh^2(\frac{\mu x}{2})\right)}. \tag{24}$$

and

$$\psi(x)_{n=3/2} = (1-\zeta) f(\zeta^{1/2}) \left(\frac{\frac{\epsilon^2+1}{\epsilon^2}}{\frac{\epsilon^2+1}{\epsilon^2} + \sinh^2(\frac{\mu x}{2})}\right) \frac{\sinh(\frac{\mu x}{2})}{\left(\frac{\epsilon^2+1}{\epsilon^2} + \sinh^2(\frac{\mu x}{2})\right)^{1/2}}. \tag{25}$$

respectively, with $\sigma(x) = \sinh(\mu x/2) \left[\frac{\epsilon^2+1}{\epsilon^2} + \sinh^2(\mu x/2)\right]^{-\frac{1}{2}}$. We note that the above is the ground state, since $P_- f(\zeta^{1/2}) = 0$, leads to

$$\left(\zeta \frac{d^2}{d\zeta^2} + \frac{1}{2}\frac{d}{d\zeta}\right) f(\zeta^{\frac{1}{2}}) = 0 \tag{26}$$

with the solution

$$f(\zeta^{\frac{1}{2}}) = 2c_1 \zeta^{\frac{1}{2}} + c_2. \tag{27}$$

Comparing this with $(\hat{H} - E) f(\zeta^{\frac{1}{2}}) = 0$, one has $c_2 = 0$, leading to the state $\psi(x)_{n=3/2}$.

It is worth pointing out, that we limited (2) to only 3 singularities, which led to an algebra with three generators $\{P_+, P_0, P_-\}$. Extension of this algebra can provide new symmetry and more general structure. Assuming $f_i(x)$ to be $(i = 1, \ldots, 3)$,

$$f_1(x) = \sum_{n=0}^{N} a_n x^n, \qquad f_2(x) = \sum_{n=0}^{N-1} b_n x^n, \qquad f_3(x) = \sum_{n=0}^{N-2} c_n x^n$$

we get operators of order (N-2) and the algebra extends to $\{P_{N-2}, P_{N-1}, \ldots, P_0, P_{-1}, P_{-2}\}$. Note that generalized and tri-confluent Heun differential equation can be treated as examples of this extension [28, 29]. These algebraic structures need careful study, which is currently under investigation.

In conclusion, we have investigated the algebraic structure of a wide class of differential equations connected with spectral problems. It is found that, these equations, naturally possess deformed $sl(2)$ as their underlying symmetry algebra. The hidden symmetry of Heun differential equation is shown to be the cubic deformation of $sl(2)$ algebra, whereas the confluent and bi-confluent Heun lead to quadratic deformation. It is explicitly shown that exactly solvable and QES systems emerge from our analysis in an unified manner. We also explicitly show the algebraic structure of a dynamical systems not connected with the $sl(2)$ structure. In future, we would like to extend this investigation to other equations like tri-confluent and generalized Heun and generalize the study of this algebraic structure to higher dimensional symmetries.

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